The Factorization method applied to cracks with impedance boundary conditions

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Abstract

We use the Factorization method to retrieve the shape of cracks with impedance boundary conditions from farfields associated with incident plane waves at a fixed frequency. This work is an extension of the study initiated by Kirsch and Ritter [Inverse Problems, 16, pp. 89-105, 2000] where the case of sound soft cracks is considered. We address here the scalar problem and provide theoretical validation of the method when the impedance boundary conditions hold on both sides of the crack. We then deduce an inversion algorithm and present some validating numerical results in the case of simply and multiply connected cracks.

1 Introduction

This work is concerned with the reconstruction of the shape of cracks with impedance boundary conditions in a homogeneous background from acoustic measurements using the so-called Factorization method [15]. The considered data for this inverse problem is formed by the farfields associated with incident plane waves at a fixed frequency. As a sampling method, the Factorization method gives a simple and fast algorithm. In addition, it has the advantage, compared to other sampling methods [4, 21, 8], of giving an exact characterization of the crack using the behavior of an indicator function. However, this characterization is proved only for a restricted set of the impedance values (see main theorem below).

This work is an extension of the study initiated in [16] (see also [6] for the electrostatic case) where the case of sound soft cracks is addressed. We consider here the case where impedance boundary conditions hold on both sides of the crack.

Two difficulties on the theoretical level have to be faced. The first one is due to the fact that the far field operator is no longer normal when the impedance values are not real. Therefore, only the second version of the factorization method (the so called $F_\sharp$ version) can be considered. To do so, a slight modification of the main theoretical tool behind this method has been introduced (see also [18]). The second one is related to the geometrical nature of the crack problem. For instance as opposed to the case of an obstacle with a non empty interior [12, 15], a two by two boundary operator is needed in the factorization of the far field operator. As we shall indicate later, this requires in particular a non intuitive choice of the unknowns and the functional spaces associated with. We shall discuss two possible choices of the factorization. The obtained main result requires unusual assumptions on the real part of the sum of the two impedances, as we impose this quantity to be positive definite on the crack.

The case of impedance boundary conditions also causes difficulties in designing the numerical algorithm associated with the theoretical part. The latter requires a proper choice of the orientation of the probing "small" crack. In order to fix this problem a minimization procedure with respect to the normal (similar to [4]) is incorporated in the definition of the indicator function.

For an overview of recent works on other sampling methods applied to crack identification for the Helmholtz equation with (different) impedance boundary conditions we refer to [9, 3, 21, 4, 22] and the references therein. The litterature on sound soft or sound hard crack identification for the Helmholtz equation is rich and we refer to [13, 1, 14, 17] and references therein for an account of different non linear inversion method in the case of single/few measurements and to [5, 2] and references therein for the case of small cracks.

The remainder of the paper is organized as follows. In Section 2, we briefly present the forward problem and some key results on variational solutions to this problem. Section 3 introduces the inverse problem and the main theoretical result associated with the factorization method. In Section 4 we discuss two different choices of the farfield factorizations and prove the auxiliary results required by the proof of our main theorem. The last section is dedicated to the presentation of the numerical algorithm associated with the theoretical result together with some validating examples in the case of single and multiply connected cracks.

2 The forward scattering problem by an impedance crack

We start this section by introducing the direct scattering problem from an impedance crack in a homogeneous medium. Let $\sigma \subset \mathbf{R}^m$, $m = 2, 3$, be a smooth nonintersecting open arc. For further considerations, we assume that σ can be extended to an arbitrary smooth, simply connected, closed curve $\partial\Omega$ enclosing a bounded domain Ω in $\mathbf{R}^m$. The normal vector ν on σ coincides with the outward normal vector to $\partial\Omega$.
Impedance type boundary conditions on σ leads to the following problem

$$\begin{cases} \Delta u + \kappa^2 u = 0 & \text{in } \mathbf{R}^m \setminus \sigma, \\ \partial_\nu u_\pm \pm \lambda^\pm u_\pm = 0 & \text{on } \sigma, \end{cases} \tag{1}$$

where the wave number κ is positive and $\lambda^\pm \in L^\infty(\sigma)$ are the given (complex-valued) impedance functions with non-negative imaginary part. We used the notation $u_\pm(x) := \lim_{h\to 0^+} u(x \pm h\nu)$ and $\partial_\nu u_\pm(x) := \lim_{h\to 0^+} \nu \cdot \nabla u(x \pm h\nu)$ for $x \in \sigma$ (for regular functions u) and also will use the short notation $[u] := u_+ - u_-$ and $[\partial_\nu u] := \partial_\nu u_+ - \partial_\nu u_-$ on σ.
The total field $u = u^i + u^s$ is decomposed into the given incident plane wave $u^i(x, d) = e^{i\kappa d\cdot x}$ with unitary direction d and the unknown scattered field u^s which is required to satisfy the Sommerfeld radiation condition

$$\lim_{r=|x|\to+\infty} r^{\frac{m-1}{2}} (\partial_r u^s - i\kappa u^s) = 0, \tag{2}$$

uniformly in all directions $\hat{x} = \frac{x}{|x|}$. In order to formulate the scattering problem more precisely we need to define the trace spaces on σ. If $H^{1/2}(\partial\Omega)$ and $H^{-1/2}(\partial\Omega)$ denote the usual Sobolev spaces on the closed regular curve $\partial\Omega$, we introduce the following spaces

$$\begin{array}{l} H^{1/2}(\sigma) := \left\{ u_{|\sigma} : \, u \in H^{1/2}(\partial\Omega) \right\}, \\ \tilde{H}^{1/2}(\sigma) := \left\{ u \in H^{1/2}(\partial\Omega) : \, supp(u) \subset \bar{\sigma} \right\}, \end{array}$$

and we denote by $H^{-1/2}(\sigma)$ and $\tilde{H}^{-1/2}(\sigma)$ the dual spaces of $\tilde{H}^{1/2}(\sigma)$ and $H^{1/2}(\sigma)$ respectively (see [19]). We notice that one has the inclusions

$$\tilde{H}^{1/2}(\sigma) \subset H^{1/2}(\sigma) \subset L^2(\sigma) \subset \tilde{H}^{-1/2}(\sigma) \subset H^{-1/2}(\sigma).$$

Introducing $g^{\pm} = -(\partial_\nu \pm \lambda^{\pm})u^i$ the problem (1)-(2) can be seen as a special case of what will be referred to as **Impedance Crack Problem (ICP)**: Find $u^s \in H^1_{loc}(\mathbb{R}^m \setminus \sigma)$ satisfying the Sommerfeld radiation condition (2) and

$$\begin{array}{rcll} \Delta u^s + \kappa^2 u^s & = & 0 & \text{in } \mathbf{R}^m \setminus \sigma, \\ \partial_\nu u^s_{\mp} \pm \lambda^{\pm} u^s_{\pm} & = & g^{\pm} & \text{on } \sigma, \end{array}$$

where $H^1_{loc}(\mathbb{R}^m \setminus \sigma)$ denotes the space of functions that belong to $H^1_{loc}(B)$ for all bounded set $B \subset \mathbb{R}^m \setminus \sigma$. We recall that u^s has the asymptotic behavior ([10])

$$u^s(x) = \gamma \frac{e^{i\kappa r}}{r^{(m-1)/2}}(u_\infty(\hat{x}) + O(1/r)),$$

where the function u_∞ is called the farfield pattern. The constant $\gamma = \frac{e^{i\pi/4}}{\sqrt{8k\pi}}, \frac{1}{4\pi}$ respectively for $m = 2$ and $m = 3$.

We shall discuss hereafter the class of data $g^{\pm}$ for which existence can be ensured, which will be crucial in the design of the inversion method. Motivated by later use, we chose to adopt a variational approach in the study of ICP.

Denote by B_R a sufficiently large ball with radius R containing $\bar{\sigma}$ and by S_R its boundary. We introduce $T_R : H^{1/2}(S_R) \to H^{-1/2}(S_R)$, the Dirichlet to Neumann operator, defined by

$$T_R(\varphi) = \partial_r w, \qquad \text{on} \quad S_R, \tag{3}$$

with $w \in H^1_{loc}(\mathbf{R}^m \setminus B_R) \cap H^2_{loc}(\mathbf{R}^m \setminus \overline{B_R})$ being the unique solution satisfying the Sommerfeld radiation condition and verifying

$$\left\{ \begin{array}{rcll} \Delta w + k^2 w & = & 0 & \text{in } \mathbf{R}^m \setminus \overline{B}_R, \\ w & = & \varphi & \text{on } S_R. \end{array} \right.$$

Let $\langle , \rangle_{S_R}$ denotes the duality product between $H^{-1/2}(S_R)$ and $H^{1/2}(S_R)$ that coincides with $L^2(S_R)$ scalar product for regular functions. We recall that (see for instance [20, 11]),

$$\Re \left\langle T_R \varphi, \varphi \right\rangle_{S_R} < 0 \text{ and } \Im \left\langle T_R \varphi, \varphi \right\rangle_{S_R} > 0 \quad \forall \varphi \in H^{1/2}(S_R), \varphi \neq 0. \tag{4}$$

Assume for a moment that $g^{\pm}$ are in $L^2(\sigma)$. Then, (using standard proofs for the use of the operator T_R in variational formulations [20]) u^s is a solution of ICP if and only if $u^s \in H^1(B_R \setminus \sigma)$ and satisfies for all $v \in H^1(B_R \setminus \sigma)$

$$\int_{B_R \setminus \sigma} (\nabla u^s \overline{\nabla v} - \kappa^2 u^s \bar{v}) dx - \int_\sigma (\lambda^+ u^s_+ \overline{v_+} + \lambda^- u^s_- \overline{v_-}) ds - \left\langle T_R(u^s_{|S_R}), v \right\rangle_{S_R} \\ = - \int_\sigma (g^+ \overline{v_+} - g^- \overline{v_-}) ds. \tag{5}$$

By denoting $l(v)$ the right hand side of (5) we remark that existence of solution to this variational formulation requires the continuity of l on $H^1(B_R \setminus \sigma)$. Using classical trace theorems, we have that $v_\pm \in H^{1/2}(\sigma)$ and therefore the antilinear form l is continuous if $g^\pm$ are in $L^2(\sigma)$ and also if $g^\pm$ are in $\tilde{H}^{-1/2}(\sigma)$, where in the latter case the integrals have to be understood as duality pairing $\tilde{H}^{-1/2}(\sigma)$–$H^{1/2}(\sigma)$. As it will be clearer later, this is not enough to be able to analyze the Factorization method. One needs in particular to enlarge the class of admissible data. This can be done by observing that

$$l(v) = -\int_\sigma g^+\overline{[v]}ds - \int_\sigma (g^+ - g^-)\overline{v_-}ds. \tag{6}$$

Therefore, since for $v \in H^1(B_R\setminus\sigma)$ the jump $[v] \in \tilde{H}^{1/2}(\sigma)$, the antilinear form l is still continuous on $H^1(B_R\setminus\sigma)$ if we simply assume that $g^\pm \in H^{-1/2}(\sigma)$ and $(g^+-g^-) \in \tilde{H}^{-1/2}(\sigma)$, where in that case the integrals in (6) have to be respectively understood as $H^{-1/2}(\sigma)$–$\tilde{H}^{1/2}(\sigma)$ and $\tilde{H}^{-1/2}(\sigma)$–$H^{1/2}(\sigma)$ duality pairings. We then have the existence of a constant C_R independent from $g^\pm$ such that

$$|l(v)| \leq C_R\|v\|_{H^1(B_R\setminus\sigma)} \left(\|g^+\|_{H^{-1/2}(\sigma)} + \|g^+ - g^-\|_{H^{-1/2}(\sigma)}\right) \quad \forall v \in B_R \setminus \sigma.$$

Lemma 2.1. *Assume that $g^\pm \in H^{-1/2}(\sigma)$ such that $(g^+ - g^-) \in \tilde{H}^{-1/2}(\sigma)$. Then, (ICP) has a unique solution that continuously depends on the boundary data g^+ and g^-.*

Proof. The only remaining part is to prove that the operator associated with the right hand side of (5), denoted by $A(u^s, v)$, is invertible. Since this is a classical exercise we shall give here only an outline of the proof. We first prove that it is a Fredholm operator of index 0 by decomposing A into a coercive part

$$A_0(u^s, v) := \int_{B_R\setminus\sigma} (\nabla u^s\overline{\nabla v} + u^s\bar{v})dx - \left\langle T_R(u^s_{|S_R}), v\right\rangle_{S_R}, \tag{7}$$

and a compact one

$$B(u^s, v) := -(\kappa^2 + 1)\int_{B_R\setminus\sigma} u^s\bar{v}dx - \int_\sigma (\lambda^+u^s_+\overline{v_+} + \lambda^-u^s_-\overline{v_-})ds. \tag{8}$$

The coercivity of A_0 directly follows from the first property of T_R in (4) while the compactness of B follows from trace theorems and the Rellich compact embedding theorem. We then prove the injectivity of the operator by taking the imaginary part of $A(u^s, u^s) = 0$, which implies, using the sign assumption on the imaginary parts of $\lambda^\pm$,

$$Im\left\langle T_R(u^s_{|S_R}), u^s|S_R\right\rangle_{S_R} = 0.$$

Therefore, using the second property in (4) and the definition of T_R, we obtain $(u^s_{|S_r}, \partial_\nu u^s_{|S_R}) = (0, 0)$. Hence, using a standard unique continuation argument, u^s vanishes in $B_R \setminus \sigma$. □

3 Statement of the Inverse Problem and the Main Theorem

Let us denote by $u^\infty(\cdot, d)$ the farfield associated with (ICP) solutions corresponding to $g^\pm = -(\partial_\nu\pm\lambda^\pm)u^i(\cdot, d)_{|\sigma}$. Our inverse problem consists in reconstructing σ from the knowledge of $u^\infty(\cdot,\cdot)$ on $\mathbf{S}^{m-1}\times\mathbf{S}^{m-1}$. As stated before, we shall employ the factorization method to solve this inverse problem. For that purpose we define the farfield operator

$$\begin{array}{rccl} F: & L^2(\mathbf{S}^{m-1}) & \to & L^2(\mathbf{S}^{m-1}) \\ & g & \mapsto & \displaystyle\int_{\mathbf{S}^{m-1}} u^\infty(\cdot,d)g(d)ds(d) \end{array} \tag{9}$$

and introduce for any smooth non intersecting open arc $L\subset\mathbf{R}^m$, the farfields $\Phi_L^\infty\in L^2(\mathbf{S}^{m-1})$ defined by

$$\Phi_L^\infty(\hat{x}) := \int_L \left(-i\kappa\;\hat{x}.\nu(y)\alpha_L(y)+\beta_L(y)\right)e^{-i\kappa\hat{x}\cdot y}ds(y), \tag{10}$$

with densities $\alpha_L\in\tilde{H}^{\frac{1}{2}}(L)$ and $\beta_L\in\tilde{H}^{-\frac{1}{2}}(L)$. A characterization of σ is obtained using the solvability of

$$(F_\sharp)^{1/2}(g_L)(\hat{x}) = \Phi_L^\infty(\hat{x}) \qquad\text{for all}\quad \hat{x}\in\mathbf{S}^{m-1} \tag{11}$$

where $F_\sharp$ is defined by

$$F_\sharp := |\Re F| + \Im F,$$

$\Re F := \frac{1}{2}(F+F^*)$, and $\Im F := \frac{1}{2i}(F-F^*)$. The main theorem is the following.

Theorem 3.1 (Main Theorem)**.** *Further assume that $(\lambda^+ + \lambda^-)^{-1}\in L^\infty(\sigma)$ and there exists a constant $c>0$ such that*

$$\Re(\lambda^+ + \lambda^-) \geq c|\lambda^+ + \lambda^-|^2 \quad \textit{a.e. on } \sigma.$$

Then, for any smooth non intersecting arc L and functions $\alpha_L\in\tilde{H}^{\frac{1}{2}}(L)$ and $\beta_L\in L^2(L)$ such that the support of (α_L,β_L) is $\bar{L}$, the following is true:

$$L\subset\sigma \qquad \textit{if and only if} \qquad \Phi_L^\infty\in Range(F_\sharp^{1/2}). \tag{12}$$

We recall that (12) is verified if and only if

$$\sum_{n=1}^{\infty}\frac{|\,(\Phi_L^\infty,\psi_n)_{L^2(S^2)}\,|^2}{\lambda_n} < +\infty$$

where $\{\lambda_n,\psi_n\}_{n\in\mathbf{N}}$ is an eigensystem of the self-adjoint, positive and compact operator $F_\sharp$. The proof of this theorem is based on the following abstract result which is a slight modification of the classical one stated in [15] and can also be seen as a particular case of the one used in [18]. For the reader convenience, we included a proof of this theorem into an appendix based on the original proof in [15].

Theorem 3.2. *Let $H\subset U\subset H^*$ be a Gelfand triple with a Hilbert space U and a reflexive Banach space H such that the embedding is dense. Moreover, let Y be a second Hilbert space and let $F:Y\longrightarrow Y, \mathcal{H}:Y\longrightarrow H$, and $\;T:H\longrightarrow H^*$ be linear bounded operators such that*

$$F = \mathcal{H}^*T\mathcal{H}. \tag{13}$$

We make the following assumptions:
(A1) $\mathcal{H}^*$ is compact with dense range.

(A2) $\Re[T] = C + K$ *with some compact operator* K *and some self-adjoint and coercive operator* $C : H \longrightarrow H^*$*, i.e, there exists* $c > 0$ *with*

$$\langle \varphi, C\varphi \rangle \geq \|\varphi\|^2 \qquad \textit{for all} \qquad \varphi \in H, \tag{14}$$

and $\Im T$ *is positive on the closure of the range of* $\mathcal{H}$*. Then the operator* $F_\sharp = |\Re F| + \Im F$ *is positive, and the ranges of* $\mathcal{H}^* : H^* \longrightarrow Y$ *and* ${F_\sharp}^{1/2} : Y \longrightarrow Y$ *coincide.*

The goal of next section is to provide a factorization of F in the form (13) and verifying the assumptions of Theorem 3.2. Based on the results of that section we can already give the proof of Theorem 3.1.

Proof of Theorem 3.1. A factorization of F in the form (13) is given by (28) with the space $H = H^{-1/2}(\sigma) \times L^2(\sigma)$. The assumptions required for the operator $\mathcal{H}^*$ are verified in Lemma 4.3 and those required for the operator T are verified in Lemma 4.2 and Lemma 4.1. The result of our theorem is then a consequence of Theorem 3.2 and Lemma 4.4. $\square$

4 Factorizations of the far field operator

As explained above this section is concerned with the proof of the theoretical ingredients needed in the proof of our main theorem. The first ingredient is the factorization of the operator F as in (13). There are multiple possible factorizations and we shall present here two of them. The first one, called "natural factorization" is the one inspired by the writing of ICP equations and therefore the first one would think of. We shall explain however why this factorization causes difficulties in the space function settings of the forward problem and what would be the correct setting. The latter was suggested by a second factorization for which the analysis is more simple to present. We shall give the proofs only for the second factorization, which is enough for our main result (Theorem 3.2). The proofs for the first factorization can be easily deduced using (21) .

4.1 A natural factorization

By linearity of the forward problem with respect to the incident wave, $F(\varphi)$ is nothing but the farfield of ICP solution u^s_φ associated with $g^\pm = -(\partial_\nu v_\varphi \pm \lambda^\pm v_\varphi)$ where v_φ is the Herglotz wave of kernel φ defined by

$$v_\varphi(x) := \int_{\mathbf{S}^{m-1}} e^{i\kappa x\cdot d} \varphi(d)\, ds(d), \quad x \in \mathbf{R}^m.$$

As suggested by the study of the forward problem we define

$$H_1 := \{(g^+, g^-) \in H^{-1/2}(\sigma) \times H^{-1/2}(\sigma) \text{ such that } (g^+ - g^-) \in \tilde{H}^{-1/2}(\sigma)\}$$

and consider $\mathcal{G}_1 : H_1 \rightarrow L^2(\mathbf{S}^{m-1})$ that maps the boundary data (g^+, g^-) to the farfield pattern of the solution ICP. Introducing $\mathcal{H}_1 : L^2(\mathbf{S}^{m-1}) \rightarrow H_1$ the operator defined by

$$\mathcal{H}_1(\varphi) := (-(\partial_\nu + \lambda^+) v_{\varphi|\sigma}, -(\partial_\nu - \lambda^-) v_{\varphi|\sigma}), \tag{15}$$

then we immediately get

$$F = \mathcal{G}_1 \mathcal{H}_1.$$

For $(\alpha,\beta)\in L^2(\sigma)\times L^2(\sigma)$, we observe that after changing the order of integration,

$$\begin{aligned}
&\int_\sigma \mathcal{H}_1(\varphi)\cdot\overline{(\alpha,\beta)}ds\\
&=-\int_{\mathbf{S}^{m-1}}\varphi(d)\overline{\left[\int_\sigma\left(\alpha(y)\left(\frac{\partial}{\partial\nu(y)}+\overline{\lambda^+}\right)e^{-i\kappa d\cdot y}+\beta(y)\left(\frac{\partial}{\partial\nu(y)}-\overline{\lambda^-}\right)e^{-i\kappa d\cdot y}\right)ds(y)\right]}ds(d)\\
&=-\int_{\mathbf{S}^{m-1}}\varphi(d)\overline{\left[\int_\sigma\left((\alpha(y)+\beta(y))\frac{\partial e^{-i\kappa d\cdot y}}{\partial\nu(y)}+\left(\overline{\lambda^+}\alpha(y)-\overline{\lambda^-}\beta(y)\right)e^{-i\kappa d\cdot y}\right)ds(y)\right]}ds(d).
\end{aligned}$$

We therefore deduce that the adjoint operator $\mathcal{H}_1^*$ is given by

$$\mathcal{H}_1^*(\alpha,\beta)(\hat{x})=-\int_\sigma\left((\alpha(y)+\beta(y))\frac{\partial e^{-i\kappa\hat{x}\cdot y}}{\partial\nu(y)}+\left(\overline{\lambda^+}\alpha(y)-\overline{\lambda^-}\beta(y)\right)e^{-i\kappa\hat{x}\cdot y}\right)ds(y),\qquad(16)$$

for $\hat{x}\in\mathbf{S}^{m-1}$. We now recall that (using the Green representation formula of scattered fields [10]), the farfield of the ICP solution can be represented as

$$\mathcal{G}_1(g^+,g^-)(\hat{x})=\int_\sigma\left([u^s](y)\frac{\partial e^{-i\kappa\hat{x}\cdot y}}{\partial\nu(y)}-[\partial_\nu u^s](y)e^{-i\kappa\hat{x}\cdot y}\right)ds(y).\qquad(17)$$

Consequently, considering $T_1\,:\,(g^+,g^-)\mapsto(\alpha,\beta)$ such that

$$\alpha+\beta=-[u^s]\quad\text{and}\quad\overline{\lambda^+}\alpha-\overline{\lambda^-}\beta=[\partial_\nu u^s]\quad\text{on }\sigma,\qquad(18)$$

where u^s is the solution to ICP, we simply get $\mathcal{G}_1=\mathcal{H}_1^*T_1$ which finally leads to the factorization

$$F=\mathcal{H}_1^*T_1\mathcal{H}_1.\qquad(19)$$

Simple algebra shows that (18) is equivalent to

$$\alpha=\frac{1}{\overline{\lambda^++\lambda^-}}[\partial_\nu u^s]-\frac{\overline{\lambda^-}}{\overline{\lambda^++\lambda^-}}[u^s]\quad\text{and}\quad\beta=-\frac{1}{\overline{\lambda^++\lambda^-}}[\partial_\nu u^s]-\frac{\overline{\lambda^+}}{\overline{\lambda^++\lambda^-}}[u^s].$$

Therefore the operator T_1 is defined on H_1 by

$$T_1(g^+,g^-)=\left(\frac{1}{\overline{\lambda^++\lambda^-}}[\partial_\nu u^s]-\frac{\overline{\lambda^-}}{\overline{\lambda^++\lambda^-}}[u^s],-\frac{1}{\overline{\lambda^++\lambda^-}}[\partial_\nu u^s]-\frac{\overline{\lambda^+}}{\overline{\lambda^++\lambda^-}}[u^s]\right),\qquad(20)$$

where u^s is the solution to ICP. The issue with this factorization is that one cannot prove continuity of $T_1:H_1\to H_1^*$: Using the simple relation

$$\int_\sigma(f^+g^++f^-g^-)ds=\int_\sigma f^+(g^+-g^-)ds+\int_\sigma(f^++f^-)g^-ds$$

one easily identify

$$H_1^*=\{(f^+,f^-)\in H^{1/2}(\sigma)\times H^{1/2}(\sigma)\text{ such that }(f^++f^-)\in\tilde{H}^{1/2}(\sigma)\}.$$

Due to the presence of the normal derivative in the expression of T_1 one can easily deduce our claim. It turned out however, that if one replaces H_1 with the smaller subspace

$$\tilde{H}_1:=\{\{(g^+,g^-)\in H^{-1/2}(\sigma)\times H^{-1/2}(\sigma)\text{ such that }(g^+-g^-)\in L^2(\sigma)\}$$

where its dual is defined as

$$\tilde{H}_1^*=\{(f^+,f^-)\in L^2(\sigma)\times L^2(\sigma)\text{ such that }(f^++f^-)\in\tilde{H}^{1/2}(\sigma)\},$$

then $T_1\,:\,\tilde{H}_1\to\tilde{H}_1^*$ is continuous. One can also prove that factorization (19) fits into the framework of Theorem 3.2. These facts were suggested to us by (and can be easily deduced from) the second factorization presented below.

4.2 A second factorization

This factorization is based on rewriting ICP in terms of the boundary data

$$h^+ = g^+ \text{ and } h^- = g^+ - g^-. \tag{21}$$

We then consider for a given boundary data (h^+, h^-), the scattered field $\tilde{u}^s \in H^1_{loc}(\mathbb{R}^m \setminus \sigma)$ satisfying the Sommerfeld radiation condition (2) and

$$\begin{cases} \Delta \tilde{u}^s + \kappa^2 \tilde{u}^s = 0 & \text{in } \mathbf{R}^m \setminus \overline{\sigma}, \\ \partial_\nu \tilde{u}^s_+ + \lambda^+ \tilde{u}^s_+ = h^+ & \text{on } \sigma, \\ [\partial_\nu \tilde{u}^s] + \lambda^+ \tilde{u}^s_+ + \lambda^- \tilde{u}^s_- = h^- & \text{on } \sigma. \end{cases} \tag{22}$$

It is obvious that $\tilde{u}^s$ coincides with the solution of ICP if (h^+, h^-) and (g^+, g^-) are related by (21). We then deduce that (22) is well posed for $(h^+, h^-) \in H^{-1/2}(\sigma) \times \tilde{H}^{-1/2}(\sigma)$. As indicated above, we shall (up to making stronger assumptions on the impedances later on) restrict our selves to $(h^+, h^-) \in H_2$ where

$$H_2 := H^{-1/2}(\sigma) \times L^2(\sigma).$$

We observe, for later use, that solving (22) is equivalent to solve the following variational problem: $\tilde{u}^s \in H^1(B_R \setminus \sigma)$ and satisfy for all $v \in H^1(B_R \setminus \sigma)$,

$$\int_{B_R \setminus \sigma} (\nabla \tilde{u}^s \overline{\nabla v} - \kappa^2 \tilde{u}^s \bar{v}) dx - \int_\sigma (\lambda^+ \tilde{u}^s_+ \overline{v_+} + \lambda^- \tilde{u}^s_- \overline{v_-}) ds - \left\langle T_R(\tilde{u}^s_{|S_R}), v \right\rangle_{S_R}$$
$$= - \left\langle h^+, [v] \right\rangle_{H^{-1/2}(\sigma), \tilde{H}^{1/2}(\sigma)} - \int_\sigma h^- \overline{v_-} ds. \tag{23}$$

Proceeding as in the previous factorization, we first introduce $\mathcal{G}_2 : H_2 \to L^2(\mathbf{S}^{m-1})$ that maps the boundary data (h^+, h^-) to the farfield pattern associated with $\tilde{u}^s$ solution of (22) and the operator $\mathcal{H}_2 : L^2(\mathbf{S}^{m-1}) \to H_2$ defined by

$$\mathcal{H}_2(\varphi) = \left(-(\partial_\nu + \lambda^+) v_{\varphi|\sigma}, -(\lambda^- + \lambda^+) v_{\varphi|\sigma} \right), \tag{24}$$

where v_φ is the Herglotz wave with kernel $\varphi \in L^2(\mathbf{S}^{m-1})$. One then immediately get that $F = G_2 \mathcal{H}_2$. If we denote by $\mathcal{H}_2^*$ the adjoint of the operator $\mathcal{H}_2$, then similar calculations as for $\mathcal{H}_1^*$ show that $\mathcal{H}_2^*$ is given by

$$\mathcal{H}_2^*(\alpha, \beta)(\hat{x}) = -\int_\sigma \alpha(y) \frac{\partial e^{-i\kappa \hat{x} \cdot y}}{\partial \nu(y)} ds(y) - \int_\sigma \left(\bar{\lambda}^+ \alpha(y) + \overline{(\lambda^+ + \lambda^-)} \beta(y) \right) e^{-i\kappa \hat{x} \cdot y} ds(y), \tag{25}$$

for $\hat{x} \in \mathbf{S}^{m-1}$. We also recall that the farfield associated with $\tilde{u}^s$ can be represented as

$$\mathcal{G}_2(g^+, g^-)(\hat{x}) = \int_\sigma \left([\tilde{u}^s](y) \frac{\partial e^{-i\kappa \hat{x} \cdot y}}{\partial \nu(y)} - [\partial_\nu \tilde{u}^s](y) e^{-i\kappa \hat{x} \cdot y} \right) ds(y). \tag{26}$$

Therefore, considering $T_2 : (g^+, g^-) \mapsto (\alpha, \beta)$ such that

$$\alpha = -[\tilde{u}^s] \qquad \text{and} \qquad \bar{\lambda}^+ \alpha + \overline{(\lambda^+ + \lambda^-)} \beta = [\partial_\nu \tilde{u}^s], \quad \text{on } \sigma, \tag{27}$$

where $\tilde{u}^s$ is the solution of (22), we simply get $G_2 = \mathcal{H}_2^* T_2$, which finally leads to the second factorization

$$F = \mathcal{H}_2^* T_2 \mathcal{H}_2. \tag{28}$$

Simple algebra shows that (27) is equivalent to

$$\alpha = -[\tilde{u}^s] \quad \text{and} \quad \beta = \frac{h^-}{\lambda^+ + \lambda^-} - \frac{2i\Im(\lambda^+)}{\lambda^+ + \lambda^-}\tilde{u}^s_+ - \frac{\lambda^- + \bar{\lambda}^+}{\lambda^+ + \lambda^-}\tilde{u}^s_-.$$

Therefore T_2 is defined by

$$T_2(h^+, h^-) = (-[\tilde{u}^s], \frac{h^-}{\lambda^+ + \lambda^-} - \frac{2i\Im(\lambda^+)}{\lambda^+ + \lambda^-}\tilde{u}^s_+ - \frac{\lambda^- + \bar{\lambda}^+}{\lambda^+ + \lambda^-}\tilde{u}^s_-) \tag{29}$$

where $\tilde{u}^s \in H^1_{loc}(\mathbf{R}^m \setminus \sigma)$ is the solution to (22). Since $H_2^* = \tilde{H}^{1/2}(\sigma) \times L^2(\sigma)$ we immediately get from the expression (29) that, if we further assume that $(\lambda^+ + \lambda^-)^{-1} \in L^\infty(\sigma)$, then $T_2 : H_2 \to H_2^*$ is bounded.

4.3 Analysis of the second factorization (28)

This section is dedicated to the proof of the auxiliary results that served to the proof of our main theorem. We shall denote by $\langle , \rangle$ the $H_2 - H_2^*$ duality product that extends the $L^2(\sigma) \times L^2(\sigma)$ scalar product.

Lemma 4.1. *Further assume that $(\lambda^+ + \lambda^-)^{-1} \in L^\infty(\sigma)$. Then the operator $T_2 : H_2 \to H_2^*$ is bounded and satisfy for all $(h^+, h^-) \in H_2$, $(h^+, h^-) \neq 0$,*

$$\Im \big\langle (h^+, h^-), T_2(h^+, h^-) \big\rangle < 0.$$

Proof. The continuity of T_2 is an obvious consequence of the well posedness of (22) and trace theorems. From the expression of T_2 one easily get

$$\begin{aligned}\big\langle (h^+, h^-), T_2(h^+, h^-) \big\rangle = &- \big\langle h^+, [\tilde{u}^s] \big\rangle_{H^{-1/2}(\sigma), \tilde{H}^{1/2}(\sigma)} \\ &+ \int_\sigma \frac{|h^-|^2}{\lambda^+ + \lambda^-} ds - \int_\sigma \frac{2i\Im(\lambda^+) h^-}{\lambda^+ + \lambda^-}\overline{\tilde{u}^s_+} ds - \int_\sigma h^- \frac{\lambda^+ + \bar{\lambda}^-}{\lambda^+ + \lambda^-}\overline{\tilde{u}^s_-} ds,\end{aligned}$$

that we rewrite

$$\begin{aligned}\big\langle (h^+, h^-), T_2(h^+, h^-) \big\rangle = &- \big\langle h^+, [\tilde{u}^s] \big\rangle_{H^{-1/2}(\sigma), \tilde{H}^{1/2}(\sigma)} - \int_\sigma h^- \overline{\tilde{u}^s_+} \\ &+ \int_\sigma \frac{|h^-|^2}{\lambda^+ + \lambda^-} ds - \int_\sigma \frac{2i\Im(\lambda^+) h^-}{\lambda^+ + \lambda^-}\overline{\tilde{u}^s_+} ds - \int_\sigma \frac{2i\Im(\lambda^-) h^-}{\lambda^+ + \lambda^-}\overline{\tilde{u}^s_-} ds.\end{aligned}$$

Therefore, using the variational formulation (22) with $v = \tilde{u}^s$ we get

$$\begin{aligned}\big\langle (h^+, h^-), T_2(h^+, h^-) \big\rangle = &\int_{B_R \setminus \sigma} (|\nabla \tilde{u}^s|^2 - \kappa^2 |\tilde{u}^s|^2) dx - \int_\sigma (\lambda^- |\tilde{u}^s_-|^2 + \lambda^+ |\tilde{u}^s_+|^2) ds - \langle T_R \tilde{u}^s, \tilde{u}^s \rangle_{S_R} \\ &+ \int_\sigma \frac{|h^-|^2}{\lambda^+ + \lambda^-} ds - \int_\sigma \frac{2i\Im(\lambda^+) h^-}{\lambda^+ + \lambda^-}\overline{\tilde{u}^s_+} ds - \int_\sigma \frac{2i\Im(\lambda^-) h^-}{\lambda^+ + \lambda^-}\overline{\tilde{u}^s_-} ds.\end{aligned}$$

Consequently, taking the imaginary part gives

$$\Im \left\langle (h^+, h^-), T_2(h^+, h^-) \right\rangle = -\int_\sigma (\Im(\lambda^-)|\tilde{u}^s_-|^2 + \Im(\lambda^+)|\tilde{u}^s_+|^2) ds - \Im \langle T_R \tilde{u}^s, \tilde{u} \rangle_{S_R}$$
$$- \int_\sigma \frac{\Im(\lambda^+ + \lambda^-)|h^-|^2}{|\lambda^+ + \lambda^-|^2} - \int_\sigma 2\Im(\lambda^+)\Re(\frac{h^-}{\lambda^+ + \lambda^-}\overline{\tilde{u}^s_+}) - \int_\sigma 2\Im(\lambda^-)\Re(\frac{h^-}{\lambda^+ + \lambda^-}\overline{\tilde{u}^s_-})$$

Rearranging the terms in front of $\Im(\lambda^-)$ and $\Im(\lambda^+)$, one finally get the identity

$$\Im \left\langle (h^+, h^-), T_2(h^+, h^-) \right\rangle = -\int_\sigma \Im(\lambda^-) \left| \tilde{u}^s_- + \frac{h^-}{\lambda^+ + \lambda^-} \right|^2 - \Im \langle T_R(\tilde{u}^s), \tilde{u}^s \rangle_{S_R}$$
$$- \int_\sigma \Im(\lambda^+) \left| \tilde{u}^s_+ + \frac{h^-}{\lambda^+ + \lambda^-} \right|^2 .$$

Now suppose that $\Im \langle (h^+, h^-), T_2(h^+, h^-) \rangle = 0$, then

$$\Im \langle T_R(\tilde{u}^s), \tilde{u}^s \rangle = 0,$$

which implies $(\partial_\nu \tilde{u}^s, \tilde{u}^s) = (0, 0)$ on S_R. Using a standard unique continuation argument we obtain that $\tilde{u}^s$ vanishes on $B_R \setminus \sigma$ and therefore $(h^+, h^-) = (0, 0)$. $\square$

Lemma 4.2. *Further assume that $(\lambda^+ + \lambda^-)^{-1} \in L^\infty(\sigma)$ and there exists a constant $c > 0$ such that*

$$\Re(\lambda^+ + \lambda^-) \geq c|\lambda^+ + \lambda^-|^2 \quad \text{a.e. on } \sigma.$$

Then, the operator $T_2 : H_2 \to H_2^*$ can be decomposed as $T_2 = T_2^0 + T_2^c$ where $T_2^c : H_2 \to H_2^*$ is a compact operator and $T_2^0 : H_2 \to H_2^*$ is defined by

$$T_2^0(h^+, h^-) = (-[u^0], \frac{h^-}{\overline{\lambda^+ + \lambda^-}} - u^0_-), \tag{30}$$

with $u^0 \in H^1(B_R \setminus \sigma)$ the solution of the variational problem:

$$\int_{B_R \setminus \sigma} (\nabla u^0 \overline{\nabla v} + u^0 \bar{v}) dx - \left\langle T_R u^0, v \right\rangle_{S_R} = - \left\langle h^+, [v] \right\rangle_{H^{-1/2}(\sigma), \tilde{H}^{1/2}(\sigma)} - \int_\sigma h^- \overline{v_-} ds \tag{31}$$

for all $v \in H^1(B_R \setminus \sigma)$. Moreover $C := \Re(T_2^0)$ satisfies the coercivity property (14) on $H = H_2$.

Proof. We first verify the coercivity property for C. From the expression of T_2^0

$$\left\langle (h^+, h^-), T_2^0(h^+, h^-) \right\rangle = - \left\langle h^+, [u^0] \right\rangle_{H^{-1/2}(\sigma), \tilde{H}^{1/2}(\sigma)} - \int_\sigma h^- \overline{u^0_-} ds + \int_\sigma \frac{|h^-|^2}{\lambda^+ + \lambda^-} ds,$$

and from the variational problem (31), with $v = u^0$,

$$\int_{B_R \setminus \sigma} (|\nabla u^0|^2 + |u^0|^2) dx - \left\langle T_R u^0, v \right\rangle_{S_R} = - \left\langle h^+, [u^0] \right\rangle_{H^{-1/2}(\sigma), \tilde{H}^{1/2}(\sigma)} - \int_\sigma h^- \overline{u^0_-} ds.$$

Consequently, using the first property in (4) and the assumptions on $\lambda^+ + \lambda^-$, we get the existence of a positive constant c such that

$$\Re \left\langle (h^+, h^-), T_2^0(h^+, h^-) \right\rangle \geq c \left(\|u^0\|^2_{H^1(B_R \setminus \sigma)} + \|h^-\|^2_{L^2(\sigma)} \right). \tag{32}$$

The variational solution u^0 of (31) satisfies.

$$\begin{cases} \Delta u^0 - u^0 & = & 0 & \text{in } B_R \setminus \overline{\sigma}, \\ \partial_\nu u^0_+ & = & h^+ & \text{on } \sigma, \\ \partial_\nu u^0_- & = & (h^+ - h^-) & \text{on } \sigma, \\ \partial_\nu u^0 - T_R(u^0) & = & 0 & \text{on } S_R. \end{cases} \tag{33}$$

Therefore, using the definition of $H^{-1/2}(\sigma)$ and trace theorems,

$$\|h^+\|^2_{H^{-1/2}(\sigma)} \leq \|\partial_\nu u^0_+\|^2_{H^{-1/2}(\partial\Omega)} \leq K(\|\Delta u^0\|^2_{L^2(\Omega)} + \|\nabla u^0\|^2_{L^2(\Omega)}) \leq K\|u^0\|^2_{H^1(B_R\setminus\sigma)},$$

for some constant K. Combined with (32), this inequality proves the desired coercivity property.
We now prove that the operator $T_2^c = T_2 - T_2^0$ is a compact. We first observe

$$\begin{array}{rlcl} T_2^c: & H_2 & \to & H_2^* \\ & (h^+, h^-) & \mapsto & (-[w], -\dfrac{2i\Im(\lambda^+)}{\overline{\lambda^+ + \lambda^-}}\tilde{u}^s_+ - \dfrac{2i\Im(\lambda^-)}{\overline{\lambda^+ + \lambda^-}}\tilde{u}^s_- + u^0_-) \end{array} \tag{34}$$

where $w := \tilde{u}^s - u^0$, $\tilde{u}^s \in H^1(B_R \setminus \sigma)$ is the solution of (22) and $u^0 \in H^1(B_R \setminus \sigma)$ is the solution of (31). The compactness of the second component of T_2^c is then a simple consequence of the continuity of the solutions to (22) and (31), the trace theorem and the Rellich compact embedding theorem. For the first component of T_2^c we simply observe that $w \in H^1(B_R \setminus \sigma)$ satisfy

$$A_0(w, v) = -B(\tilde{u}^s, v) \quad \text{for all } v \in H^1(B_R \setminus \sigma),$$

where A_0 and B are respectively defined by (7) and (8). Since A_0 is coercive and B defines a compact operator on $H^1(B_R \setminus \sigma)$, the mapping $\tilde{u}^s \mapsto w$ is compact from $H^1(B_R \setminus \sigma)$ into $H^1(B_R \setminus \sigma)$. The desired result then follows from the continuity of the solution to (22) and trace theorem.

□

Lemma 4.3. *Further assume that $(\lambda^+ + \lambda^-)^{-1} \in L^\infty(\sigma)$. Then the operator $\mathcal{H}_2^* : H_2^* \to L^2(\mathbf{S}^{m-1})$ defined by (25) is one to one and has a dense range.*

Proof. Let $(\alpha, \beta) \in H_2^*$ such that $\mathcal{H}_2^*(\alpha, \beta) = 0$. We note that $\mathcal{H}_2^*$ is the far field pattern of the potential

$$V(x) = \int_\sigma (-\alpha(y)\partial_{\nu(y)}\Phi(x, y) + (\overline{\lambda^+}\alpha(y) - (\overline{\lambda^+ + \lambda^-})\beta(y))\Phi(x, y))ds(y),$$

for $x \in \mathbf{R}^m \setminus \bar{\sigma}$, where Φ denotes the Green function associated with the Helmholtz equation and satisfying the Sommerfeld radiation condition. Therefore, if $\mathcal{H}_2^*(\alpha, \beta) = 0$, then from Rellich's lemma and the unique continuation principle, we conclude that $V = 0$ in $\mathbf{R}^m \setminus \bar{\sigma}$. By the jump properties of the layer potentials ([19]), we have $[V] = -\alpha$ and $[\partial_\nu V] = -\overline{\lambda^+}\alpha(y) + (\overline{\lambda^+ + \lambda^-})\beta(y)$. This implies

$$\alpha = 0 \qquad \text{and} \qquad -\overline{\lambda^+}\alpha(y) + (\overline{\lambda^+ + \lambda^-})\beta(y) = 0.$$

Finally, since by assumption $(\lambda^+ + \lambda^-)^{-1} \in L^\infty(\sigma)$, $\alpha = \beta = 0$ and proves that $\mathcal{H}^*$ is one to one.

In order to prove the density of the range of the operator $\mathcal{H}^*{}_2$, we shall prove the injectivity of its adjoint $\mathcal{H}_2$. Let $g \in L^2(S^{m-1})$ be an element of the kernel of $\mathcal{H}_2$. Then,

$$(-\partial_\nu + \lambda^+)v_g = 0 \qquad \text{and} \qquad (\lambda^+ + \lambda^-)v_g = 0 \qquad\qquad \text{on} \quad \sigma.$$

Since by assumption $(\lambda^+ + \lambda^-)^{-1} \in L^\infty(\sigma)$ then $v_g = 0$ and $\partial_\nu v_g = 0$ on $\sigma_0 \subset \sigma$. From a unique continuation argument, this implies that $v_g = 0$ in $\mathbf{R}^m$ and therefore $g = 0$ ([10]), which proves the desired result. □

Lemma 4.4. *Further assume that $(\lambda^+ + \lambda^-)^{-1} \in L^\infty(\sigma)$. Then, for any smooth non intersecting arc L and functions $\alpha_L \in \tilde{H}^{\frac{1}{2}}(L)$, $\beta_L \in L^2(L)$ such that the support of (α_L, β_L) is $\bar{L}$, the function Φ_L^∞ given by (10) belongs to Range$(\mathcal{H}_2^*)$ if and only if $L \subset \sigma$.*

Proof. First assume that $L \subset \sigma$. We define α as the extension of $-\alpha_L$ by 0 to all σ, which indeed gives a function in $\tilde{H}^{1/2}(\sigma)$. We then define β as $\beta = 0$ on $\sigma \setminus L$ and $\beta = -(\beta_L + \bar{\lambda}^+\alpha_L)/(\bar{\lambda}^+ + \bar{\lambda}^-)$ on L, which indeed provides a function in $L^2(\sigma)$. We then easily verify from expressions (25) and (10) that $\mathcal{H}_2^*(\alpha, \beta) = \Phi_L^\infty$.

Now let $L \not\subset \sigma$ and assume, on the contrary, that $\Phi_L^\infty \in Range(\mathcal{H}^*{}_2)$. Hence, there exists $\varphi \in \tilde{H}^{1/2}(\sigma)$ and $\psi \in L^2(\sigma)$ such that

$$\Phi_L^\infty(\hat{x}) = \int_L \left(-\varphi(y)\frac{\partial e^{-i\kappa\hat{x}\cdot y}}{\partial\nu(y)} - (\overline{\lambda^+}\varphi(y) + \overline{(\lambda^+ + \lambda^-)}\psi(y))e^{-i\kappa\hat{x}\cdot y}(y) \right) ds(y).$$

Thus Φ_L^∞ is the far field pattern of the potential

$$P(x) = \int_\sigma \left(-\varphi(y)\partial_{\nu(y)}\Phi(x,y) - (\overline{\lambda^+}\varphi(y) + \overline{(\lambda^+ + \lambda^-)}\psi(y))\Phi(x,y) \right) ds(y), \quad x \in \mathbf{R}^m \setminus \overline{\sigma}.$$

Since by definition Φ_L^∞ is also the far field pattern of the potential Φ_L given by

$$\Phi_L(x) = \int_L (\alpha_L(y)\partial_{\nu(y)}\Phi(x,y) + \beta_L(y)\Phi(x,y))ds(y), \quad x \in \mathbf{R}^m \setminus \bar{L}, \tag{35}$$

then using the Rellich lemma and the unique continuation principle, the potentials Φ_L and P coincide in $\mathbf{R}^m \setminus (\bar{\sigma} \cup \bar{L})$.
Let $x_0 \in L$ and B_ϵ a small neighborhood of x_0 such that $B_\epsilon \cap \sigma = \emptyset$ and $B_\epsilon \cap L$ has a non empty interior in L. Then $P \in H^2(B_\epsilon)$. Consequently, Φ_L and its normal derivatives are continuous across $B_\epsilon \cap L$. Using jump properties of layer potentials, this implies that $(\alpha_L, \beta_L) = 0$ on $B_\epsilon \cap L$, which contradicts the fact that the support of (α_L, β_L) is $\bar{L}$. □

5 Numerical algorithm and results

The numerical experiments are conducted in a 2D setting of the problem. We consider n equally distant observation points of the farfield $(\hat{x}_l)_{1\le l\le n}$ on the unit circle. We then approximate the farfield operator using the trapezoidal rule

$$Fg(\hat{x}_l) \simeq \frac{2\pi}{n}\sum_{j=1}^{n} u_\infty(\hat{x}_l, \hat{x}_j)g(\hat{x}_j) \tag{36}$$

The farfields $u_\infty(\cdots, \hat{x}_j)$ are generated synthetically by solving the forward problem using an integral equation approach [4]. This data is then corrupted with a pointwise random noise: $u_\infty(\hat{x}_l, \hat{x}_j) = u_\infty^{synth}(\hat{x}_l, \hat{x}_j)(1+\epsilon(r_1+ir_2))$ where u_∞^{synth} is the computed data, r_1 and r_2 are two random numbers in the interval $[-1, 1]$ and where ϵ is the noise level. In all our experiments $\epsilon = 0.01$ and we used (without trying to optimize this number) $n = 100$.
Let L be a small segment of center z and with normal ν. Then we approximate Φ_L^∞ by

$$\Phi_L^\infty(\hat{x}_l) \simeq \gamma |L| (-i\kappa\ \hat{x}_l.\nu\ \alpha(z) + \beta(z)) e^{-i\kappa \hat{x}_l z}.$$

The discrete equation to solve is then

$$(F_\sharp)^{1/2} g_L(\hat{x}_l) \simeq \Phi_L^\infty(\hat{x}_l) \qquad \forall \hat{x}_l. \tag{37}$$

Using a regularization by truncation, the solution of (37) is given by

$$\|g_L\|^2_{L^2(S^1)} \simeq \sum_{k=1}^{M} \frac{|\,(\Phi_L^\infty, \psi_k)_{L^2(S^1)}\,|^2}{\lambda_k} \tag{38}$$

where $\{\lambda_n, \psi_n\}_{n\in\mathbf{N}}$ is an eigensystem of the self-adjoint and positive operator $F_\sharp$ and M is a regularization parameter (fixed in the numerical experiments by trial and error). We shall consider two types of solutions. The first one denoted by g_z corresponds to $\alpha(z) = 1$ and $\beta(z) = 0$ whereas the second one denoted by $g_{z,\nu}$ corresponds to $\alpha(z) = 0$ and $\beta(z) = 1$.
The normal ν can be expressed

$$\nu = \zeta\nu_1 + \sqrt{1-\zeta^2}\nu_2$$

with $-1 \le \zeta \le 1$ and where $\nu_1 := (0, 1)$ and $\nu_2 := (1, 0)$. Therefore, by linearity of equation (37),

$$g_{z,\nu} = \zeta g_{z,\nu_1} + \sqrt{1-\zeta^2} g_{z,\nu_2}.$$

According to Theorem 3.1, if $z \in \sigma$ and the normal ν does not coincides with a normal to σ at z, then $\|g_{z,\nu}\|^2$ goes to infinity as $M \to \infty$ while it remains bounded in the opposite case. We therefore expect for $z \in \sigma$, the min value with respect to ζ of

$$\|g_{z,\nu}\|^2 = \zeta^2 \|g_{z,\nu_1}\|^2 + (1-\zeta^2)\|g_{z,\nu_2}\|^2 + 2\zeta\sqrt{1-\zeta^2}\,\langle g_{z,\nu_1}\ , g_{z,\nu_2}\rangle \tag{39}$$

to remain bounded as $M \to \infty$. According to Theorem 3.1, the min value of (39) becomes unbounded as $M \to \infty$ if $z \notin \sigma$. As an indicator function of the crack location we then propose

$$z \to \frac{1}{\|g_z\|} + \frac{1}{\|g_{z,\nu}\|},$$

where $g_{z,\nu}$ corresponds with ζ that minimizes (39).

Figures 1, 2, 3 and 4 show the isolines of this indicator function for different shapes of the cracks and different values of the impedances. The wavelength being equal to 1, we choose three different shapes:

- (a) a broken segment with vertices $(0, 0.8)$, $(0, 0)$ and $(0.4, -0.8)$,
- (b) an arc centered at $(-0.5, 0.5)$ with radius 0.75 and angle $\theta \in [0,\ \pi/2]$,
- (c) an L shape with vertices $(0.75, 0), (0, 0)$ and $(0, 0.75)$.

We observe that a very good reconstruction of the crack is obtained when the impedance values are relatively small or relatively large. The case of impedances with "intermediate" values is less accurate but still provide a good approximation of the crack shape and location. These conclusions are in concordance with those reported in [4] for the case of the Linear Sampling Method, although slightly better reconstructions are obtained using the Factorization method for the case of impedances with intermediate values. The last example, represented in Figure 5 show the obtained reconstructions for a multiply connected crack made of two segments and two arcs. The results demonstrate the capability of the method to handle this complex configuration, even with different values of the impedances associated with each crack.

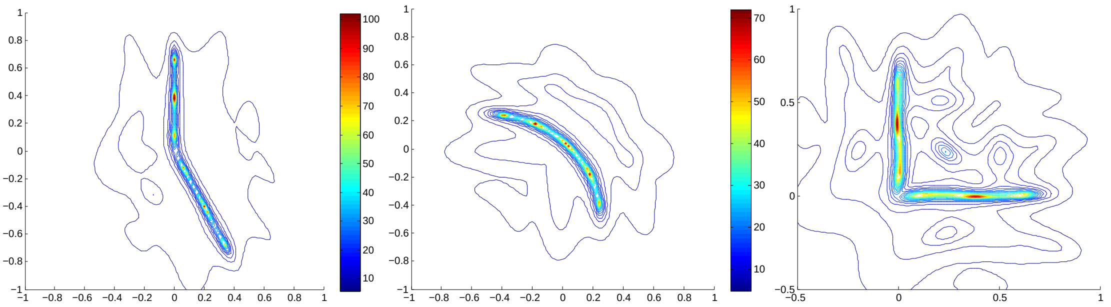

Figure 1: Reconstruction of cracks for $\lambda^- = \lambda^+ = 10^{-2}(1+i)$. Left: geometry (a), middle: geometry (b), right: geometry (c). Wavelength = 1 and random noise level = 1%.

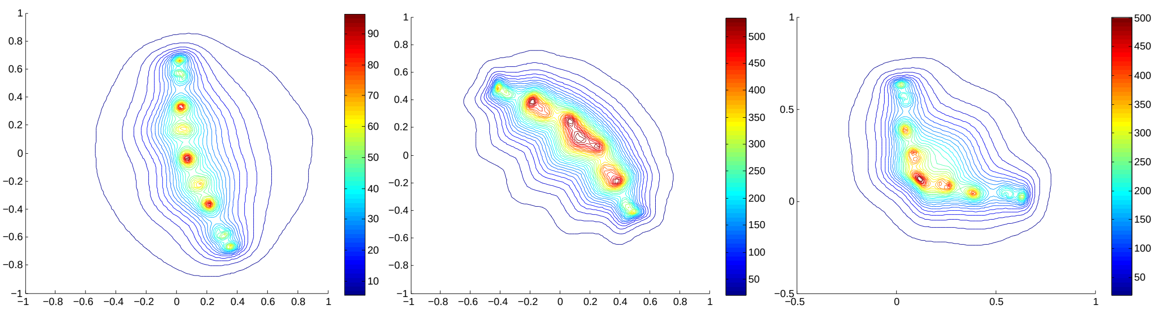

Figure 2: Reconstruction of cracks for $\lambda^- = \lambda^+ = 5+5i$. Left: geometry (a), middle: geometry (b), right: geometry (c). Wavelength = 1 and random noise level = 1%.

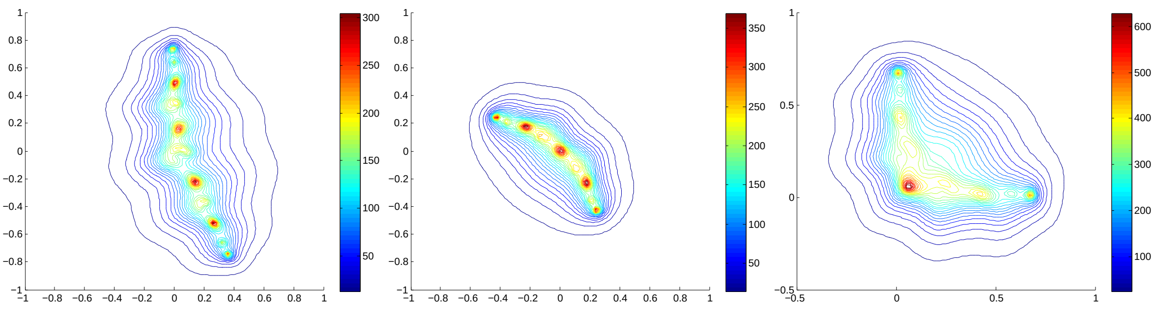

Figure 3: Reconstruction of cracks for $\lambda^- = \lambda^+ = 10 + 10i$. Left: geometry (a), middle: geometry (b), right: geometry (c). Wavelength = 1 and random noise level = 1%.

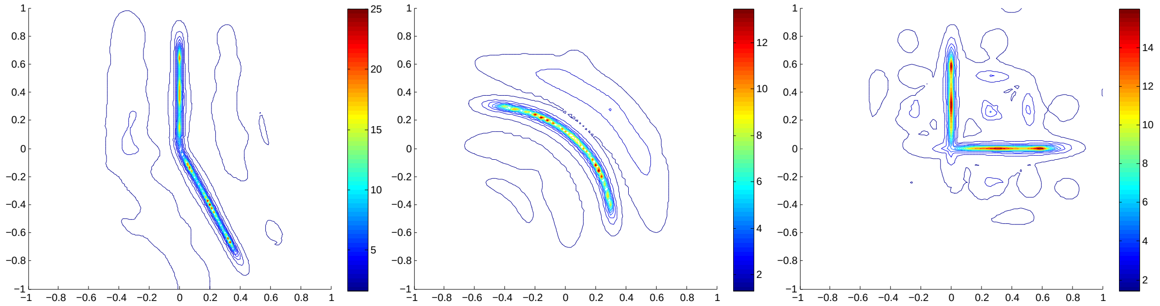

Figure 4: Reconstruction of cracks for $\lambda^- = \lambda^+ = 10^2(1+i)$. Left: geometry (a), middle: geometry (b), right: geometry (c). Wavelength = 1 and random noise level = 1%.

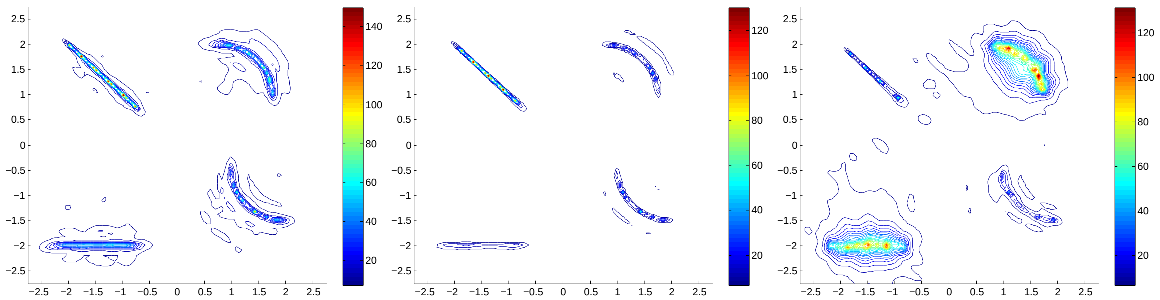

Figure 5: Reconstruction of multiple cracks numbered 1 (segment top-left), 2 (arc top-right), 3 (segment bottom-left) and 4 (arc bottom-right). Figure left: $\lambda_1^\pm = \lambda_2^\pm = \lambda_3^\pm = \lambda_4^\pm = 10^2(1+i)$. Figure middle: $\lambda_1^\pm = \lambda_2^\pm = \lambda_3^\pm = \lambda_4^\pm = 10^{-2}(1+i)$. Figure right: $\lambda_1^\pm = \lambda_4^\pm = 10^{-2}$ and $\lambda_3^\pm = \lambda_2^\pm = 10(1+i)$. Wavelength = 1 and random noise level = 1%.

A Proof of Theorem 3.2

This theorem is a slight modification of [15, Theorem 2.15, p57] since we do not require $\Im T$ to be compact. We shall explain hereafter why the compactness of $\Im T$ is not needed in the proof of [15, Theorem 2.15, p57]. This compactness is only used in part E of that proof to show that the operator $T_\sharp$ (using the notation of [15]) is Fredholm of index zero and is invertible. We shall indicate how the latter is still true if $\Im T$ is not compact, but still a positive operator.

Part A of the proof of [15, Theorem 2.15, p57] shows that we can restrict ourselves to the case where $\mathcal{H}^*$ is injective, $C = I$ and $H = U$. Parts B, C and D show that $T_\sharp : U \to U$ is a bounded (self-adjoint) operator and can be expressed as

$$T_\sharp := \Re T(Q^+ - Q^-) + \Im T$$

where Q^+ and Q^- are bounded projectors that satisfy: $Q^+ + Q^- = I$, $Q^+ - Q^-$ is an isomorphism and Q^- has finite rank. Moreover the operator $\Re T(Q^+ - Q^-)$ is non negative.

Using the decomposition $\Re T = C + K$ we can decompose $T_\sharp$ as follows.

$$T_\sharp = C(Q^+ + Q^-) - 2CQ^- + K(Q^+ - Q^-) + \Im T = C + \Im T - 2CQ^- + K(Q^+ - Q^-)$$

Since C is coercive and $\Im T$ is non negative, we infer that $C + \Im T$ is coercive. Since K is compact and $Q^+ - Q^-$ bounded, $K(Q^+ - Q^-)$ is also compact. Q^- has a finite rank, hence $2CQ^-$ is also compact. These prove that $T_\sharp$ is a Fredholm operator with index zero. Now, since $\Re T(Q^+ - Q^-)$ is non negative and $\Im T$ is positive, this proves that $T_\sharp$ is injective and therefore has a bounded inverse $T_\sharp^{-1}$.

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